

ON THE SURVIVAL PROBABILITY IN THE MATHERON - DE MARSILY MODEL

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ABSTRACT. We are interested in the behaviour of the range and of the first return time to the origin of random walks in random scenery. As a byproduct a precise estimate of the survival probability in the Matheron and de Marsily model [18] is obtained. Our result confirms the conjectures announced in [17, 19].

1. RESULTS FOR RANDOM WALKS IN RANDOM SCENERY

Random walks in random scenery (RWRS) are simple models of processes in disordered media with long-range correlations. They have been used in a wide variety of models in physics to study anomalous dispersion in layered random flows [18], diffusion with random sources, or spin depolarization in random fields (we refer the reader to Le Doussal's review paper [14] for a discussion of these models).

On the mathematical side, motivated by the construction of new self-similar processes with stationary increments, Kesten and Spitzer [13] and Borodin [3, 4] introduced RWRS in dimension one and proved functional limit theorems. This study has been completed in many works, in particular in [1] and [8]. These processes are defined as follows. Let $\xi := (\xi_y, y \in \mathbb{Z})$ and $X := (X_k, k \geq 1)$ be two independent sequences of independent identically distributed random variables taking their values in \mathbb{Z} . The sequence ξ is called the *random scenery*. The sequence X is the sequence of increments of the *random walk* $(S_n, n \geq 0)$ defined by $S_0 := 0$ and $S_n := \sum_{i=1}^n X_i$, for $n \geq 1$. The *random walk in random scenery* (RWRS) Z is then defined by

$$Z_0 := 0 \text{ and } \forall n \geq 1, Z_n := \sum_{k=1}^n \xi_{S_k}.$$

Denoting by $N_n(y)$ the local time of the random walk S :

$$N_n(y) := \#\{k = 1, \dots, n : S_k = y\},$$

it is straightforward to see that Z_n can be rewritten as $Z_n = \sum_y \xi_y N_n(y)$.

As in [13], the distribution of ξ_0 is assumed to belong to the normal domain of attraction of a strictly stable distribution \mathcal{S}_β of index $\beta \in (0, 2]$, with characteristic function ϕ given by

$$\phi(u) = e^{-|u|^\beta (A_1 + iA_2 \operatorname{sgn}(u))} \quad u \in \mathbb{R},$$

where $0 < A_1 < \infty$ and $|A_1^{-1}A_2| \leq |\tan(\pi\beta/2)|$. When $\beta > 1$, this implies that $\mathbb{E}[\xi_0] = 0$. When $\beta = 1$, we assume the symmetry condition $\sup_{t>0} |\mathbb{E}[\xi_0 \mathbb{1}_{\{|\xi_0| \leq t\}}]| < +\infty$.

Concerning the random walk, the distribution of X_1 is assumed to belong to the normal basin of

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attraction of a stable distribution \mathcal{S}'_α with index $\alpha \in (0, 2]$, with characteristic function ψ given by

$$\psi(u) = e^{-|u|^\alpha (C_1 + iC_2 \operatorname{sgn}(u))} \quad u \in \mathbb{R},$$

where $0 < C_1 < \infty$ and $|C_1^{-1}C_2| \leq |\tan(\pi\alpha/2)|$. In the particular case where $\alpha = 1$, we assume that $C_2 = 0$. Moreover we assume that the additive group \mathbb{Z} is generated by the support of the distribution of X_1 .

Then the following weak convergences hold in the space of càdlàg real-valued functions defined on $[0, \infty)$ endowed with the Skorohod J_1 -topology :

$$\begin{aligned} & \left(n^{-\frac{1}{\alpha}} S_{[nt]} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (Y(t))_{t \geq 0}, \\ & \left(n^{-\frac{1}{\beta}} \sum_{k=0}^{[nx]} \xi_k \right)_{x \geq 0} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (U(x))_{x \geq 0} \quad \text{and} \quad \left(n^{-\frac{1}{\beta}} \sum_{k=[-nx]}^1 \xi_k \right)_{x \geq 0} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (U(-x))_{x \geq 0} \end{aligned}$$

where $(U(x))_{x \geq 0}$, $(U(-x))_{x \geq 0}$ and $(Y(t))_{t \geq 0}$ are three independent Lévy processes such that $U(0) = 0$, $Y(0) = 0$, $Y(1)$ has distribution \mathcal{S}'_α , $U(1)$ and $U(-1)$ have distribution \mathcal{S}_β . We will denote by $(L_t(x))_{x \in \mathbb{R}, t \geq 0}$ a continuous version with compact support of the local time of the process $(Y(t))_{t \geq 0}$. Let us define

$$\delta := 1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta}.$$

In the case $\alpha \in (1, 2]$ and $\beta \in (0, 2]$, Kesten and Spitzer [13] proved the convergence in distribution of $(n^{-\delta} Z_{[nt]})_{t \geq 0}$, $n \geq 1$ (with respect to the J_1 -metric), to a process $\Delta = (\Delta_t)_{t \geq 0}$ defined in this case by

$$\Delta_t := \int_{\mathbb{R}} L_t(x) dU(x).$$

This process Δ is called Kesten-Spitzer process in the literature.

When $\alpha \in (0, 1)$ (when the random walk S is transient) and $\beta \in (0, 2] \setminus \{1\}$, $(n^{-\frac{1}{\beta}} Z_{[nt]})_{t \geq 0}$, $n \geq 1$ converges in distribution (with respect to the M_1 -metric), to $(\Delta_t := c_0 U_t)_{t \geq 0}$ for some $c_0 > 0$ (see [7]).

When $\alpha = 1$ and $\beta \in (0, 2] \setminus \{1\}$, $(n^{-\frac{1}{\beta}} (\log n)^{\frac{1}{\beta}-1} Z_{[nt]})_{t \geq 0}$, $n \geq 1$ converges in distribution (with respect to the M_1 -metric), to $(\Delta_t := c_1 U_t)_{t \geq 0}$ for some $c_1 > 0$ (see [7]).

Hence in any of the cases considered above, $(Z_{[nt]}/a_n)_{t \geq 0}$ converges in distribution (with respect to the M_1 -metric) to some process Δ , with

$$a_n := \begin{cases} n^{1-\frac{1}{\alpha}+\frac{1}{\alpha\beta}} & \text{if } \alpha \in (1, 2] \\ n^{\frac{1}{\beta}} (\log n)^{1-\frac{1}{\beta}} & \text{if } \alpha = 1 \\ n^{\frac{1}{\beta}} & \text{if } \alpha \in (0, 1). \end{cases}$$

We are interested in the asymptotic behaviour of the range \mathcal{R}_n of the RWRS Z , i.e. of the number of sites visited by Z before time n :

$$\mathcal{R}_n := \#\{Z_0, \dots, Z_n\}.$$

Remark 1. Let $\alpha \in (0, 2]$ and $\beta \in (0, 1)$. Then the RWRS is transient (see for instance [6]) and, due to an argument¹ by Derriennic [22, Lemma 3.3.27], $(\mathcal{R}_n/n)_n$ converges \mathbb{P} -almost surely to $\mathbb{P}[Z_j \neq 0, \forall j \geq 1]$.

¹We consider the ergodic dynamical system (Ω, μ, T) given by $\Omega := \mathbb{Z}^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{Z}}$, $\mu := (\mathbb{P}_{S_1})^{\otimes \mathbb{Z}} \otimes (\mathbb{P}_{\xi_1})^{\otimes \mathbb{Z}}$ and $T((\alpha_k)_k, (\epsilon_k)_k) := ((\alpha_{k+1})_k, (\epsilon_{k+\alpha_0})_k)$ (see for instance [12] for its ergodicity, p.162). We set $f((\alpha_k)_k, (\epsilon_k)_k) = \epsilon_0$. With these choices, $(Z_j)_{j \geq 1}$ has the same distribution under \mathbb{P} as $(\sum_{k=1}^j f \circ T^k)_{j \geq 1}$ under μ .

For recurrent random walks in random scenery, we distinguish the easiest case when ξ_1 takes its values in $\{-1, 0, 1\}$. In that case, $\beta = 2$, U is the standard real Brownian motion,

$$a_n = \begin{cases} n^{1-\frac{1}{2\alpha}} & \text{if } \alpha \in (1, 2] \\ \sqrt{n \log n} & \text{if } \alpha = 1 \\ \sqrt{n} & \text{if } \alpha \in (0, 1) \end{cases}$$

and the limiting process Δ is either the Kesten-Spitzer process (case $\alpha \in (1, 2]$) or the real Brownian motion (case $\alpha \in (0, 1]$). Remark that in any case the limiting process is symmetric.

Let $T_0 := \inf\{n \geq 1 : Z_n = 0\}$ be the first return time of the RWRS Z to 0.

Proposition 2. *If $\alpha \in (0, 2]$ and if ξ_1 takes its values in $\{-1, 0, 1\}$, then*

$$(1) \quad \frac{\mathcal{R}_n}{a_n} = \frac{\sup_{t \in [0,1]} Z_{[nt]} - \inf_{t \in [0,1]} Z_{[nt]} + 1}{a_n} \xrightarrow{\mathcal{L}} \sup_{t \in [0,1]} \Delta_t - \inf_{t \in [0,1]} \Delta_t.$$

Moreover

$$(2) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} = 2 \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t \right]$$

and

$$(3) \quad \lim_{n \rightarrow +\infty} \frac{n}{a_n} \mathbb{P}(T_0 > n) = \max \left(2 - \frac{1}{\alpha}, 1 \right) \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t \right].$$

The range of RWRS in the general case $\beta \in (1, 2]$ is much more delicate. Indeed, the fact that \mathcal{R}_n is less than $\sup_{t \in [0,1]} Z_{[nt]} - \inf_{s \in [0,1]} Z_{[ns]} + 1$ will only provide an upper bound; we use a separate argument to obtain the lower bound insuring that \mathcal{R}_n has order a_n .

Proposition 3. *Let $\alpha \in (0, 2]$ and $\beta \in (1, 2]$. Then*

$$(4) \quad 0 < \liminf_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \leq \limsup_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} < \infty$$

and

$$(5) \quad 0 < \liminf_{n \rightarrow +\infty} \frac{n}{a_n} \mathbb{P}(T_0 > n) \leq \limsup_{n \rightarrow +\infty} \frac{n}{a_n} \mathbb{P}(T_0 > n) < \infty$$

We actually prove that $\limsup_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \leq \mathbb{E}[\sup_{t \in [0,1]} \Delta_t - \inf_{t \in [0,1]} \Delta_t]$. The question whether $\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} = \mathbb{E}[\sup_{t \in [0,1]} \Delta_t - \inf_{t \in [0,1]} \Delta_t]$ or not is still open.

2. RESULTS FOR A TWO-DIMENSIONAL RANDOM WALK WITH RANDOMLY ORIENTED LAYERS

We are interested in the survival probability of a particle evolving on a randomly oriented lattice introduced by Matheron and de Marsily in [18] (see also [2]) to modelise fluid transport in a porous stratified medium. Supported by physical arguments, numerical simulations and comparison with the Fractional Brownian Motion, Redner [19] and Majumdar [17] conjectured that the survival probability asymptotically behaves as $n^{-\frac{1}{4}}$. In this paper we rigorously prove their conjecture. Let us describe more precisely the model and the results. Let us fix $p \in (0, 1)$. The (random) environment will be given by a sequence $\epsilon = (\epsilon_k)_{k \in \mathbb{Z}}$ of i.i.d. (independent identically distributed) centered random variables with values in $\{\pm 1\}$ and defined on the probability space

$(\Omega, \mathcal{T}, \mathbb{P})$. Given ϵ , the position of the particle M is defined as a \mathbb{Z}^2 -random walk on nearest neighbours starting from 0 (i.e. $\mathbb{P}^\epsilon(M_0 = 0) = 1$) and with transition probabilities

$$\mathbb{P}^\epsilon(M_{n+1} = (x + \epsilon_y, y) | M_n = (x, y)) = p, \quad \mathbb{P}^\epsilon(M_{n+1} = (x, y \pm 1) | M_n = (x, y)) = \frac{1-p}{2}.$$

At site (x, y) , the particle can either get down (or get up) with probability $\frac{1-p}{2}$ or move with probability p on the y 's horizontal line according to its orientation (to the right (resp. to the left) if $\epsilon_y = +1$ (resp. if $\epsilon_y = -1$)). We will write \mathbb{P} for the annealed law, that is the integration of the quenched distribution \mathbb{P}^ϵ with respect to \mathbb{P} . In the sequel this random walk will be named MdM random walk. This 2-dimensional random walk in random environment was first rigorously studied by mathematicians in [5]. They proved that the MdM random walk is transient under the annealed law \mathbb{P} and under the quenched law \mathbb{P}^ϵ for \mathbb{P} -almost every environment ϵ . It was also proved that it has speed zero. Actually the MdM random walk is closely related to RWRS. This fact was first noticed in [11]. More precisely its first coordinate can be viewed as a generalized RWRS, the second coordinate being a lazy random walk on \mathbb{Z} (see Section 5 of [6] for the details). Using this remark, a functional limit theorem was proved in [11] and a local limit theorem was established in [6], more precisely there exists some constant C only depending on p such that for n large,

$$\mathbb{P}(M_{2n} = (0, 0)) \sim Cn^{-\frac{5}{4}}.$$

Since the random walk M *does not* have the Markov property under the annealed law, we are not able to deduce the survival probability from the previous local limit theorem. Let us precise that the survival probability is the probability that the particle does not visit the y -axis (or the line $x = 0$) before time n i.e. $\mathbb{P}(T_0^{(1)} > n)$ where

$$T_0^{(1)} := \inf\{n \geq 1 : M_n^{(1)} = 0\}$$

is the first return time of the first coordinate $M^{(1)}$ of M to 0. As for RWRS the asymptotic behavior of this probability will be deduced from the range $\mathcal{R}_n^{(1)}$ of the first coordinate i.e. the number of vertical lines visited by $(M_k)_k$ up to time n , namely

$$\mathcal{R}_n^{(1)} := \#\{x \in \mathbb{Z} : \exists k = 0, \dots, n, \exists y \in \mathbb{Z} : M_k = (x, y)\}.$$

Let us recall that in [11] the first coordinate $M_{[nt]}^{(1)}$ normalized by $n^{\frac{3}{4}}$ is shown to converge in distribution to $K_p \Delta_t^{(0)}$, where $K_p := \frac{p}{(1-p)^{\frac{1}{4}}}$ and where $\Delta^{(0)}$ is the Kesten-Spitzer process Δ with U and Y two independent standard Brownian motions.

Proposition 4 (Survival probability of the MdM random walk). *$(\mathcal{R}_n^{(1)}/n^{\frac{3}{4}})_n$ converges in distribution to $K_p \left(\sup_{t \in [0,1]} \Delta_t^{(0)} - \inf_{t \in [0,1]} \Delta_t^{(0)} \right)$. Moreover*

$$(6) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n^{(1)}]}{n^{\frac{3}{4}}} = 2K_p \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t^{(0)} \right]$$

and

$$(7) \quad \lim_{n \rightarrow +\infty} n^{\frac{1}{4}} \mathbb{P}(T_0^{(1)} > n) = \frac{3}{2} K_p \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t^{(0)} \right].$$

Remark 5. *In the historical model [18], the probability p is equal to $1/3$, and in this particular case the survival probability is similar to $\kappa n^{-\frac{1}{4}}$ where*

$$\kappa = \left(\frac{3}{2^5} \right)^{1/4} \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t^{(0)} \right].$$

An open question is to give an estimation of the above expectation.

Remark 6. It is worth noticing that the range \mathcal{R}_n of the MdM random walk, i.e. the number of sites visited by M before time n : $\mathcal{R}_n := \#\{M_0, \dots, M_n\}$ is well understood. Using² again [22, Lemma 3.3.27], $(\mathcal{R}_n/n)_n$ converges \mathbb{P} -almost surely to $\mathbb{P}[M_j \neq 0, \forall j \geq 1]$, which contradicts the result announced in [16].

3. PROOFS

In this section we prove Propositions 2, 3 and 4.

Observe first that the asymptotic estimates on the tail distribution function of the first return time to the origin (3), (5), (7) are direct consequences of respective estimates (2), (4), (6) on the mean range. Indeed

$$\begin{aligned} \mathbb{E}[\mathcal{R}_n] &= 1 + \sum_{k=1}^n \mathbb{P}(Z_k \neq Z_{k-1}; \dots; Z_k \neq Z_0) \\ &= 1 + \sum_{k=1}^n \mathbb{P}(Z_1 \neq 0; \dots; Z_k \neq 0) = 1 + \sum_{k=1}^n \mathbb{P}(T_0 > k) \end{aligned}$$

by stationarity of the increments of Z under the annealed distribution. Since $(\mathbb{P}(T_0 > k))_k$ is non increasing, for every $0 < x < 1 < y$, we have

$$\frac{\mathbb{E}[\mathcal{R}_{\lfloor yn \rfloor} - \mathcal{R}_n]}{\lfloor yn \rfloor - n} \leq \mathbb{P}(T_0 > n) \leq \frac{\mathbb{E}[\mathcal{R}_n - \mathcal{R}_{\lfloor xn \rfloor}]}{n - \lfloor xn \rfloor}.$$

Hence, writing $C_- := \liminf_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n}$ and $C_+ := \limsup_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n}$, we obtain

$$\frac{y^\vartheta C_- - C_+}{y - 1} \leq \liminf_{n \rightarrow +\infty} \frac{n}{a_n} \mathbb{P}(T_0 > n) \leq \limsup_{n \rightarrow +\infty} \frac{n}{a_n} \mathbb{P}(T_0 > n) \leq \frac{C_+ - x^\vartheta C_-}{1 - x},$$

with $\vartheta := \max(1 - \frac{1}{2\alpha}, \frac{1}{2})$. This will give (3), (5); we proceed analogously for (7).

For Propositions 2, 4, we observe that $\mathcal{R}_n = \max_{0 \leq k \leq n} Z_k - \min_{0 \leq k \leq n} Z_k + 1$ and $\mathcal{R}_n^{(1)} = \max_{0 \leq k \leq n} M_k^{(1)} - \min_{0 \leq k \leq n} M_k^{(1)} + 1$ whereas for Proposition 3, we only have $\mathcal{R}_n \leq \max_{0 \leq k \leq n} Z_k - \min_{0 \leq k \leq n} Z_k + 1$. Hence the convergence of the means of the range in Propositions 2 and 4 and the upper bound for $\mathbb{E}[\mathcal{R}_n]$ in Proposition 3 will come from lemmas 7 and 8 below.

Let us start by the convergence in distribution.

Proof of the convergences in distribution. Due to the convergence for the M_1 -topology of $((a_n^{-1} Z_{\lfloor nt \rfloor})_t)_n$ to $(\Delta_t)_t$ as n goes to infinity, we know (see Section 12.3 in [21]) that $(a_n^{-1}(\max_{0 \leq k \leq n} Z_k - \min_{0 \leq \ell \leq n} Z_\ell))_n$ converges in distribution to $\sup_{t \in [0,1]} \Delta_t - \inf_{s \in [0,1]} \Delta_s$ as n goes to infinity. Due to [11], $((M_{\lfloor nt \rfloor}^{(1)} / n^{\frac{3}{4}})_t)_n$ converges in distribution to $(K_p \Delta_t^{(0)})_t$ in the Skorohod space endowed with the J_1 -metric. Hence $(n^{-\frac{3}{4}}(\max_{k=0, \dots, n} M_k^{(1)} - \min_{\ell=0, \dots, n} M_\ell^{(1)}))_n$ converges in distribution to $K_p(\sup_{t \in [0,1]} \Delta_t^{(0)} - \inf_{s \in [0,1]} \Delta_s^{(0)})$. \square

² We consider the ergodic dynamical system $(\tilde{\Omega}, \tilde{\mu}, \tilde{T})$ given by $\tilde{\Omega} := \{-1, 1\}^{\mathbb{Z}} \times \{-1, 0, 1\}^{\mathbb{Z}}$, $\tilde{\mu} := (\frac{\delta_1 + \delta_{-1}}{2})^{\otimes \mathbb{Z}} \otimes (p\delta_0 + \frac{1-p}{2}\delta_1 + \frac{1-p}{2}\delta_{-1})^{\otimes \mathbb{Z}}$ and $\tilde{T}((\epsilon_k)_k, (\omega_k)_k) = ((\epsilon_{k+\omega_0})_k, (\omega_{k+1})_k)$. We also set $\tilde{f}((\epsilon_k)_k, (\omega_k)_k) = (\epsilon_0, 0)$ if $\omega_0 = 0$, $\tilde{f}((\epsilon_k)_k, (\omega_k)_k) = (0, \omega_0)$ otherwise. We observe that $(M_j)_{j \geq 1}$ has the same distribution under \mathbb{P} as $(\sum_{k=0}^{j-1} \tilde{f} \circ \tilde{T}^k)_{j \geq 1}$ under $\tilde{\mu}$.

Lemma 7 (RWRS). *Assume $\beta > 1$, then*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\max_{k=0, \dots, n} Z_k]}{a_n} = \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t \right].$$

Lemma 8 (First coordinate of the MdM random walk).

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\max_{k=0, \dots, n} M_k^{(1)}]}{n^{\frac{3}{4}}} = K_p \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t^{(0)} \right].$$

Proof of Lemma 7. As explained above, we know that $(a_n^{-1} \max_{0 \leq k \leq n} Z_k)_n$ converges in distribution to $\sup_{t \in [0,1]} \Delta_t$ as n goes to infinity. Now let us prove that this sequence is uniformly integrable. To this end we will use the fact that, conditionally to the walk S , the increments of $(Z_n)_n$ are centered and positively associated. Let $\beta' \in (1, \beta)$ be fixed. Due to Theorem 2.1 of [10], there exists some constant $c_{\beta'} > 0$ such that

$$\begin{aligned} \mathbb{E} \left[\left| \max_{j=0, \dots, n} Z_j \right|^{\beta'} \middle| S \right] &\leq \mathbb{E} \left[\max_{j=0, \dots, n} |Z_j|^{\beta'} \middle| S \right] \\ &\leq c_{\beta'} \mathbb{E} \left[|Z_n|^{\beta'} \middle| S \right] \end{aligned}$$

so

$$\begin{aligned} \mathbb{E} \left[\left| \max_{j=0, \dots, n} Z_j \right|^{\beta'} \right] &= \mathbb{E} \left[\mathbb{E} \left[\left| \max_{j=0, \dots, n} Z_j \right|^{\beta'} \middle| S \right] \right] \\ &\leq c_{\beta'} \mathbb{E} \left[|Z_n|^{\beta'} \right]. \end{aligned}$$

It remains now to prove that $\mathbb{E}[|Z_n|^{\beta'}] = O(a_n^{\beta'})$.

Let us first consider the easiest case when the random scenery is square integrable that is $\beta = 2$, then we take $\beta' = 2$ in the above computations and observe that $\mathbb{E}[|Z_n|^2] = \mathbb{E}[\xi_0^2] \mathbb{E}[V_n]$, where V_n is the number of self-intersections up to time n of the random walk S , i.e. $V_n = \sum_x (N_n(x))^2 = \sum_{i,j=1}^n \mathbf{1}_{S_i=S_j}$. Usual computations (see Lemma 2.3 in [1]) give that

$$\mathbb{E}[V_n] = \sum_{i,j=1}^n \mathbb{P}(S_{i-j} = 0) \sim c'(a_n)^2$$

and the result follows.

When $\beta \in (1, 2)$, let us define $V_n(\beta)$ as follows

$$V_n(\beta) := \sum_{y \in \mathbb{Z}} (N_n(y))^\beta.$$

Given the random walk, Z_n is a sum of independent zero-mean random variables, then from Theorem 3 in [20], there exists some constant $C > 0$ such that for every n

$$\mathbb{E}[|Z_n|^{\beta'} \middle| S] \leq C \sum_y N_n(y)^{\beta'} \mathbb{E}[|\xi_y|^{\beta'}] \leq C V_n(\beta').$$

From which we deduce that $\mathbb{E}[|Z_n|^{\beta'}] \leq C \mathbb{E}[V_n(\beta')]$.

If $\alpha > 1$, due to Lemma 3.3 of [9], we know that $\mathbb{E}[V_n(\beta')] = O(a_n^{\beta'})$. If $\alpha \in (0, 1]$, using Hölder's inequality, we have

$$\mathbb{E}[V_n(\beta')] \leq \mathbb{E}[R_n]^{1-\frac{\beta'}{2}} \mathbb{E}[V_n]^{\frac{\beta'}{2}}.$$

Now if $\alpha = 1$, we know that $\mathbb{E}[R_n] \sim c \frac{n}{\log n}$ (see for instance Theorem 6.9, page 398 in [15]) and $\mathbb{E}[V_n] \sim cn \log n$ so $\mathbb{E}[V_n(\beta')]$ is $O\left(a_n^{\beta'}\right)$ with $a_n = n^{\frac{1}{\beta'}} (\log n)^{1-\frac{1}{\beta'}}$. In the case $\alpha \in (0, 1)$, the random walk is transient and the expectations of R_n and V_n behaves as n , we deduce that $\mathbb{E}[V_n(\beta')] = O\left(a_n^{\beta'}\right)$ with $a_n = n^{\frac{1}{\beta'}}$.

We conclude that

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\max_{j=0, \dots, n} \frac{Z_j}{a_n} \right] = \mathbb{E} \left[\max_{t \in [0, 1]} \Delta_t \right].$$

□

Proof of Lemma 8. We know that $(n^{-\frac{3}{4}} \max_{k=0, \dots, n} M_k^{(1)})_n$ converges in distribution to $K_p \sup_{t \in [0, 1]} \Delta_t^{(0)}$. To conclude, it is enough to prove that this sequence is uniformly integrable. To this end we will prove that it is bounded in L^2 .

Recall that the second coordinate of the MdM random walk is a random walk. Let us write it $(S_n)_n$. Observe that

$$M_n^{(1)} := \sum_{k=1}^n \varepsilon_{S_k} \mathbb{1}_{\{S_k = S_{k-1}\}} = \sum_{y \in \mathbb{Z}} \varepsilon_y \tilde{N}_n(y),$$

with $\tilde{N}_n(y) := \#\{k = 1, \dots, n : S_k = S_{k-1} = y\}$. Observe that \tilde{N} is measurable with respect to the random walk S and that $0 \leq \tilde{N}_n(y) \leq N_n(y)$.

Conditionally to the walk S , the increments of $(M_n^{(1)})_n$ are centered and positively associated. It follows from Theorem 2.1 of [10] that

$$\begin{aligned} \mathbb{E} \left[\left| \max_{j=0, \dots, n} M_j^{(1)} \right|^2 \middle| S \right] &\leq c_2 \mathbb{E} \left[|M_n^{(1)}|^2 \middle| S \right] \\ &\leq c_2 \sum_{y \in \mathbb{Z}} (\tilde{N}_n(y))^2 \leq c_2 V_n, \end{aligned}$$

where again $V_n = \sum_{y \in \mathbb{Z}} (N_n(y))^2$. Therefore

$$\mathbb{E} \left[\left| \max_{j=0, \dots, n} M_j^{(1)} \right|^2 \right] \leq c_2 \mathbb{E}[V_n].$$

Again the result follows from the fact that $\mathbb{E}[V_n] \sim c' n^{\frac{3}{2}}$. □

Proof of the lower bound of Proposition 3. Let $\mathcal{N}_n(x) := \#\{k = 1, \dots, n : Z_k = x\}$. Applying the Cauchy-Schwarz inequality to $n = \sum_x \mathcal{N}_n(x) \mathbb{1}_{\{\mathcal{N}_n(x) > 0\}}$, we obtain

$$n^2 \leq \sum_y \mathbb{1}_{\{\mathcal{N}_n(y) > 0\}} \sum_x (\mathcal{N}_n(x))^2 = \mathcal{R}_n \mathcal{V}_n,$$

with $\mathcal{V}_n = \sum_x (\mathcal{N}_n(x))^2 = \sum_{i,j=1}^n \mathbb{1}_{\{Z_i = Z_j\}}$ the number of self-intersections of Z up to time n and so using Jensen's inequality,

$$\frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \geq \frac{n^2}{a_n} \mathbb{E}[(\mathcal{V}_n)^{-1}] \geq \frac{n^2}{a_n} \mathbb{E}[\mathcal{V}_n]^{-1}.$$

Moreover, using the local limit theorems for the RWRS proved in [6, 7],

$$\mathbb{E}[\mathcal{V}_n] = n + 2 \sum_{1 \leq i < j \leq n} \mathbb{P}(Z_{j-i} = 0) \sim C' \frac{n^2}{a_n}.$$

Hence

$$\liminf_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \geq \frac{1}{C'} > 0.$$

□

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